INVERSE LAPLACE TRANSFORMS

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When using Laplace transforms it is usually required to invert a Laplace transform to complete a calculation. Saff and Snider give a formula for finding the inverse Laplace transform in section 8.3, in Theorem 6:

If a function $F(t)$ is piecewise smooth on every finite interval and $|F(t)|$ is bounded by $Me^{-\alpha t}$ for $t \geq 0$,then the Laplace transform $\mathcal{L}\{F\}(s)$ exists for $\Re(s) > \alpha$. For all $t > 0$ and any $a > \alpha$, the inverse transform is given by

$$\frac{F(t+)+F(t-)}{2} = \frac{1}{2\pi i}\text{p.v.}\int_{a-i\infty}^{a+i\infty} \mathcal{L}\{F\}(s)\,e^{st}ds \tag{1}$$

The quantity on the LHS copes with functions that are discontinuous at certain points. For a continuous function, the LHS would be just $F(t)$. The "p.v." stands for "principal value". The limits on the integral indicate that the range of integration extends along the vertical line $\Re(s) = a$ from $-\infty$ to ∞. In practice, we usually use Cauchy's residue theorem to evaluate the integral.

Example 1. Find the inverse transform of

$$g(s) = \frac{1}{s^2+4} \tag{2}$$

We could just look this up in a table of Laplace transforms, but we will use 1 to show how the method works.

We can write

$$g(s) = \frac{1}{(s+2i)(s-2i)} \tag{3}$$

so the function has simple poles at $s = \pm 2i$. We can choose any value of a, so to include the poles we take $a = 3$ and use the contour in Fig. 1.

We want the integral along the vertical (red) path as the limits extend to infinity. Along the curved (blue) path, we have

$$z = 3+\rho e^{i\theta} \tag{4}$$

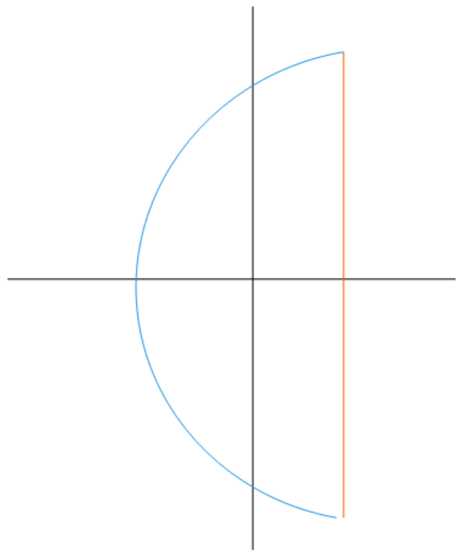

FIGURE 1. Contour for inverse Laplace transforms.

where ρ is the radius of the circle and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. We therefore have the contour integral (where we replace s by z in 1)

$$\int_C \frac{e^{(3+\rho\cos\theta)t}e^{i\rho t\sin\theta}}{(z+2i)(z-2i)}dz \tag{5}$$

Along the blue arc, for $t \geq 0$, the integral is bounded by (since $\pi\rho$ is the circumference of the arc):

$$\left|\frac{e^{(3+\rho\cos\theta)t}e^{i\rho t\sin\theta}}{z^2+4}\right|\pi\rho \leq \frac{\left|e^{(3+\rho\cos\theta)t}\right|\pi\rho}{(\rho-3)^2+4} \leq \frac{e^{3t}}{(\rho-3)^2+4}\pi\rho \to 0 \tag{6}$$

where the second inequality follows because $\cos\theta$ is negative for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Thus the contour integral becomes just the integral along the red line as $\rho \to \infty$. By the residue theorem we have

$$\int_{3-i\infty}^{3+i\infty} \frac{e^{zt}}{(z+2i)(z-2i)}dz = 2\pi i \sum_{\text{Res}} \frac{e^{zt}}{(z+2i)(z-2i)} \tag{7}$$

We can use the residue theorem to work out the residues, and we have

$$\text{Res}(2i) = \lim_{z\to 2i} \frac{e^{zt}}{z+2i} = \frac{e^{2it}}{4i} \tag{8}$$

$$\text{Res}(-2i) = \lim_{z\to -2i} \frac{e^{zt}}{z-2i} = \frac{e^{-2it}}{-4i} \tag{9}$$

Putting it together, we get

$$F(t) = \frac{1}{2\pi i}\left[2\pi i\left(\frac{e^{2it}}{4i} - \frac{e^{-2it}}{4i}\right)\right] \tag{10}$$

$$= \frac{\sin 2t}{2} \tag{11}$$

For $t < 0$, we can flip the contour in Fig. 1 horizontally about the red line, so the blue arc covers the angles $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, in which the cosine is positive. A similar analysis as above shows that the integral over the blue arc goes to zero as $\rho \to \infty$. Since the contour now contains no poles, the integral is zero, so the integral along the red line is also zero, meaning that $F(t) = 0$ if t <0.

Example 2. Find the inverse transform of

$$g(s) = \frac{4}{(s-1)^2} \tag{12}$$

We can choose $a = 2$ in 1 and use the same type of contour as before. A similar analysis shows that the integral along the blue arc goes to zero. The function now has a pole of order 2, so the residue is

$$\operatorname{Res}\left(\frac{4e^{zt}}{(z-1)^2}, 1\right) = \lim_{z\to 1}\left[\frac{d}{dz}\left((z-1)^2\frac{4e^{zt}}{(z-1)^2}\right)\right] = 4te^t \tag{13}$$

As there is only one pole, we have

$$F(t) = 4te^t \tag{14}$$

Example 3. Find the inverse transform of

$$g(s) = \frac{s+1}{s^2+4s+4} \tag{15}$$

We can write this as

$$g(s) = \frac{s+1}{(s+2)^2} \tag{16}$$

so we have single pole of order 2 at $s = -2$, with residue

$$\operatorname{Res}\left(\frac{(z+1)e^{zt}}{(z+2)^2}, -2\right) = \lim_{z\to -2}\left[\frac{d}{dz}\left((z+2)^2\frac{(z+1)e^{zt}}{(z+2)^2}\right)\right] \tag{17}$$

$$= e^{-2t}(1-t) \tag{18}$$

This is also the required inverse transform

$$F(t) = e^{-2t}(1-t) \tag{19}$$

Example 4. Find the inverse transform of

$$g(s) = \frac{1}{s^3 + 3s^2 + 2s} \tag{20}$$

$$= \frac{1}{s(s+2)(s+1)} \tag{21}$$

Here we have 3 simple poles, so we calculate the residues for each:

$$\text{Res}\left(\frac{e^{zt}}{z(z+2)(z+1)}, 0\right) = \frac{1}{2} \tag{22}$$

$$\text{Res}\left(\frac{e^{zt}}{z(z+2)(z+1)}, -2\right) = \frac{e^{-2t}}{2} \tag{23}$$

$$\text{Res}\left(\frac{e^{zt}}{z(z+2)(z+1)}, -1\right) = -e^{-t} \tag{24}$$

We therefore have

$$F(t) = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} \tag{25}$$

Example 5. Find the inverse transform of

$$g(s) = \frac{s+3}{s^2 + 4s + 7} \tag{26}$$

This one is a bit more complicated, as the denominator factors into

$$s^2 + 4s + 7 = \left(s - \left(-2 + \sqrt{3}i\right)\right)\left(s - \left(-2 - \sqrt{3}i\right)\right) \tag{27}$$

We can therefore find the residues at these two simple poles. This gets a bit messy, so using Maple to do the algebra, we get

$$\text{Res}\left(\frac{(z+3)e^{zt}}{z^2+4z+7}, -2+\sqrt{3}i\right) = -\frac{i}{6}\left(1 + i\sqrt{3}\right)\sqrt{3}\,e^{\left(-2+i\sqrt{3}\right)t} \tag{28}$$

$$\text{Res}\left(\frac{(z+3)e^{zt}}{z^2+4z+7}, -2-\sqrt{3}i\right) = \frac{i}{6}\left(1 - i\sqrt{3}\right)\sqrt{3}\,e^{\left(-2-i\sqrt{3}\right)t} \tag{29}$$

By inspection, we can see that these two residues are complex conjugates of each other, so their sum will be real. Again, using Maple to do the algebra, we find

$$\sum_{\text{Res}} = e^{-2t}\cos\left(\sqrt{3}t\right) + \frac{\sqrt{3}e^{-2t}\sin\left(\sqrt{3}t\right)}{3} \tag{30}$$

so this is the required inverse transform

$$F(t) = e^{-2t}\cos\left(\sqrt{3}t\right) + \frac{\sqrt{3}e^{-2t}\sin\left(\sqrt{3}t\right)}{3} \tag{31}$$

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